

INCOMPRESSIBLE STATES IN DOUBLE QUANTUM DOTS.

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Abstract

Incompressible (magic) states that result from many-body effects in vertically coupled quantum dots submitted to strong magnetic fields such that only the lowest Landau level is relevant are studied within an exact diagonalization calculation for $N = 3, 5$ and 6 , electrons. We find that the sequences of total angular momentum M for which these incompressible states exist depend on the interplay between the inter-dot hopping parameter Δ_t and the inter-dot distance d . For d of the order of the magnetic length and for all values of Δ_t , we conclude that, in contrast to previous claims, these incompressible states appear at magic values of M which do not differ from those obtained for a single dot, namely $M = N(N-1)/2 + jN$ where j is a positive integer number. For large inter-dot distance and simultaneously small inter-dot hopping parameter, new sequences of magic values of M are observed. These new sequences can be easily understood in terms of a transition regime towards a system of two decoupled single dots. However, important differences in the nature of the incompressible ground states are found with respect to those of a single dot.

KEYWORDS: Quantum Hall effect, double quantum dot, incompressible (magic) states.

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I. Introduction

Much effort has been devoted to understand the magic incompressible states (IS's) of two dimensional electronic nanostructures. This is due to the fact that they are closely related to the states that determine properties like superconductivity or the quantum Hall effect (QHE)^{1,2}, which are striking examples of the non-trivial behavior that strongly interacting electronic systems may display^{3,4}. Finite systems like quantum dots (QD's) provide simpler physical realizations of strongly interacting electronic systems where different models can be tested. When they are submitted to strong magnetic fields, the projection of the system to the lowest Landau level (LLL) becomes a good approximation which greatly simplifies theoretical studies in general, and, in particular, makes exact diagonalization calculations feasible. Much work has been done on single QD's in the LLL regime yielding a reasonable understanding of the nature of their IS's^{5,6}. The search of IS with well defined properties which may produce fractional QHE experimentally observable, led to analyze double layered systems⁷⁻¹². Double quantum dots (DQD's) in a vertical configuration submitted to strong magnetic fields provide a finite system in which the existence of IS is expected. However, the additional degree of freedom, together with the two new parameters, namely the distance between the dots and the tunneling strength, may give rise to new phenomenology. For instance, Yang et al.¹² suggest an experiment to test the quantum coherence of a special stable two-level system built in a DQD submitted to an adjustable interlayer bias voltage, which demonstrates suitable conditions for serving as quantum computing bits. Moreover, correlation effects can be experimentally detected in the far infrared range (FIR) using uniform electric fields with non-vanishing component along the vertical direction as the generalized Kohn theorem, under such condition does not apply¹³.

This paper is organized as follows. In Section II we describe the model used in our calculation and analyze the Hamiltonian of the system. In Section III, after the identification of the incompressible states of interacting electrons, we begin with a review of the results previously obtained for single dots and show next our main results for double dots, which cover a wide range of input parameters. Finally, in Section IV we compare our findings with previous results in the literature and draw our conclusions.

II. The Hamiltonian

We consider two identical two-dimensional quantum dots (in a vertical configuration) confined to the XY-plane by equal parabolic potentials and submitted to a strong magnetic field directed along an arbitrary direction. The Hamiltonian of the system reads,

$$H = H_0 + H_t + H_{e-e} \quad (1)$$

where H_0 is the single-particle part which contains the kinetic contribution, the confining potential and the Zeeman term. We adjust the input parameters in such a way that Landau level mixing is negligible. Then, in second quantization formalism is given by

$$H_0 = \alpha M + \beta N - \Delta_Z S \quad (2)$$

where

$$\alpha = \frac{\hbar}{2}(\sqrt{\omega_c^2 + 4\omega_0^2} - \omega_c) \quad , \quad (3)$$

$$\beta = \frac{\hbar}{2}\sqrt{\omega_c^2 + 4\omega_0^2} \quad , \quad (4)$$

and

$$\Delta_Z = \mu_B g B \quad (5)$$

ω_0 being the confining potential frequency, ω_c the cyclotron frequency given by $\omega_c = eB/m^*c$ (m^* is the effective electron mass, B the magnetic field and e and c the electron charge and the speed of light in vacuum respectively), $\mu_B = e\hbar/2mc$ the Bohr magneton and g the Landé factor (we will consider $|g| = 0.44$ whenever the Zeeman term is included). $M = \sum_{i=1}^N m_i a_{\sigma_i}^+ a_{\sigma_i}$ is the total angular momentum and N is the total number of electrons. $a_{\sigma_i}^+$ creates a single particle state and σ_i refers to the three indexes that characterize the single particle wave functions: angular momentum, spin and isospin (s or a associated to symmetric and antisymmetric combinations of wave functions concentrated in each dot: right and left). The tunneling term is given by

$$H_t = -\frac{\Delta_t}{2} X \quad (6)$$

where Δ_t is the energy gap between the symmetric and antisymmetric states in the noninteracting system and $X = N_S - N_A$ is given by the balance between symmetric and antisymmetric states. Finally the two-body interaction part of the Hamiltonian is given by

$$H_{e-e} = \frac{1}{2} \sum_{ijkl} \sum_{\Lambda=1}^3 V_{ijkl}^{(\Lambda)} a_{\sigma_i}^+ a_{\sigma_j}^+ a_{\sigma_l} a_{\sigma_k} \quad , \quad V_{ijkl}^{(\Lambda)} = \langle ij | V^{(\Lambda)} | kl \rangle \quad (7)$$

where the index Λ is used to distinguish between the three different possibilities: (i) $V^{(1)} = 0$ when only one change of a single particle isospin takes place, (ii) $V^{(2)} = \frac{1}{2}(V_{rr} + V_{rl})$ when both isospins remain unchanged and (iii) $V^{(3)} = \frac{1}{2}(V_{rr} - V_{rl})$ when both isospins are changed¹³. V_{rr} and V_{rl} are the intra and inter-dot Coulomb potentials respectively, which are given by

$$V_{rr} = \frac{e^2}{\epsilon r} \quad (8)$$

and

$$V_{rl} = \frac{e^2}{\epsilon(r^2 + d^2)^{1/2}} \quad , \quad (9)$$

d being the distance between the dots along the z -direction, \vec{r} a 2-dimensional vector and ϵ is the dielectric constant of the host semiconductor. We have assumed Dirac-delta distributions along the z -direction and have taken as a basis, Slater determinants built up from Fock-Darwin single particle wave functions projected on the LLL⁵. The diagonalization can be performed in separated subspaces characterized by three well defined quantum numbers: the total angular momentum M , the total spin S along the direction of the field \vec{B} and the parity P related to the reflexion symmetry with respect to the plane midway between the dots (P defined as $P = (-1)^{X/2}$ for even N and $P = (-1)^{(X+1)/2}$ for odd N). We will define the set (M, S, P) as a configuration.

The eigenstates within each configuration are determined by $H_{e-e} + H_t$ alone, and the role of the constant term given by H_0 is to shift the eigenenergies as a whole without changing their relative order.

III. Incompressible states in the LLL.

A. Single QD.

Before studying the IS's in DQD's, we briefly review previous work on single QD's and its consequences. For a QD an IS with total energy E and characterized by (M, S) is identified as the one which has the following singular property¹⁴: the lowest excited state with quantum numbers $(M+1, S)$ has energy $E+\alpha$. That is to say, the energetically most favorable way to excite an IS increasing its total angular momentum by one unit is by moving the system as a whole, namely by increasing by one unit the angular momentum of the center of mass (CM) only and leaving the internal structure unchanged. This characteristic was nicely recognized analyzing the Coulomb contribution to the total energy of a full polarized QD as a function of M . A periodical arrangement of plateaux (steplike structure) in the otherwise decreasing curve signaled the values of the magic angular momenta¹⁴. Furthermore, the variation of the magnetic field (or the confining potential) did not drive the ground state (GS) through all neighboring values of M but through the sequence of magic values only¹⁵.

This scenario corresponds to the regime characterized by a filling factor lower or equal to one, defined as¹⁴

$$\nu = \frac{N(N-1)}{2M} \quad (10)$$

which involves the minimum possible value of the total angular momentum for a full polarized QD given by $M_{min} = N(N-1)/2$ (the "compact state") and the angular momentum M of the magic state. Some care must be taken for low values of B for which the assumption of the LLL regime is not fulfilled. A suitable way to check this condition is by making sure that the energy of the highest single-particle occupied state is much smaller than $\omega_+ = \frac{\hbar}{2}(\sqrt{\omega_c^2 + 4\omega_0^2} + \omega_c)$, which is the energy gap between Landau levels for non-interacting electrons. The filling factor refers to the number of sublevels occupied within the LLL. There are two sublevels (spin up and down) in the case of a single QD

and four (two for spin and two for isospin) in the case of a DQD. In general, for regimes in which several sublevels are occupied, the filling factor of a QD is not well defined. GS's which are not related with IS are also possible under such multiple-sublevel occupancy.

The sequence of magic filling factors depend on the number of electrons, for $N = 3$ the values of ν are $\nu = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$ or for $N = 4$ they are $\nu = 1, \frac{3}{5}, \frac{3}{7}, \frac{1}{3}, \dots$, in both cases related to the magic angular momentum given by

$$M = \frac{1}{2}N(N - 1) + jN \quad (11)$$

where j is a positive integer number. It turns out that the analysis of the Coulomb contribution to the total energy as a function of M gives exhaustive and precise information about the magic values of the angular momentum and hence about the magic filling factors. The magic values of M are the initial values of the plateaux. However, no information about the total spin of the IS's comes from the previous analysis. In the $N = 3$ case, for a QD, the sequence of GS's is always full polarized ($S_z = 3/2$) if the Zeeman term is included in the Hamiltonian (with $|g| = 0.44$) or in contrast, oscillations between $S_z = 3/2$ and $S_z = 1/2$ were obtained if no Zeeman term is included in the calculation^{14,15}. However, in the last case, the changes in spin and angular momentum do not appear simultaneously.

B. DQD for $d \sim l_B$.

For a DQD we have a richer parameter space to be explored as, in addition to the parameters of a single QD, Δ_t and d also enter the Hamiltonian, which open new possibilities for IS's to exist. We will focus on the phase diagram (Δ_t / d) for standard values of the remaining input parameters. Due to the fact that Coulomb interaction and changes in parity are coupled processes in a DQD, we define the "interaction" energy as the Coulomb plus the tunneling contribution (C+T).

For $d \sim l_B$, where l_B is the magnetic length given by $l_B = \sqrt{\frac{\hbar}{m^*(\omega_c^2 + 4\omega_0^2)^{1/2}}}$, the pure Coulomb contribution to the total energy ($\alpha = \beta = \Delta_t = \Delta_Z = 0$) as a function of M , is a decreasing function without plateaux as it is shown in the curves (a) in Fig.1 and 2 for $N = 5$ and $N = 6$ respectively (energies are given in units of $u = e^2/\epsilon l_B$). Figs.1A and 2A correspond to parity $P = 1$ and Figs.1B and 2B to $P = -1$. All four cases refer to full polarized systems ($S=5/2$ for $N=5$ and $S=3$ for $N=6$). For each value of M , the energy displayed is the lowest within the configuration (M, S, P) . The absence of plateaux can be understood as follows. Since $\Delta_t = 0$, the number of electrons in each dot is a well defined number. Hence, in order to increase the total angular momentum by one unit, the angular momentum of either dot must be increased by one unit, which unavoidably increases the typical distance from the electrons of one dot to the ones of the other dot, and, therefore, decreases the inter-dot Coulomb energy.

According to Figs. 1 and 2, it is necessary to include a sizeable tunneling contribution in order to obtain a sequence of plateaux, which, furthermore, only occur for $P = -1$.

Indeed, from a series of calculations for $N = 5$ (not shown in Figs.1), which correspond to a variation of Δ_t from 0 to 2 *meV* by small steps, we see a number of plateaux gradually appearing as Δ_t increases. We find that from $\Delta_t = 2$ *meV* to 0.8 *meV* the sections from $M = 10$ to $M = 11$ and from $M = 15$ to $M = 16$ are exact plateaux. For $\Delta_t = 0.4$ *meV* they are approximately flat and for $\Delta_t = 0.2$ *meV* they disappear. However in all cases the curves are abruptly decreasing before $M = 10$ and between magic values. That is to say, we do not find any extra value of magic M different from those given by Eq.(11).

In order to obtain information about the spin and parity of the IS's, we calculated the Coulomb plus tunneling contribution for all possible configurations. Fig.3 (for $N = 5$) shows that the sequence of plateaux appear only when the system is fully polarized in spin and have parity $P = -1$ (similar results were obtained for $N = 6$). Furthermore, although the parity $P = -1$ for $N=5$ can be obtained from different values of X , i.e., $X = N_S - N_A = 5, 1$ or -3 , the occupancy of the single-particle states for such incompressible GS turns out to be $X = 5$ only, namely the system is always full spin and isospin polarized. This suggests that the GS that are IS's will not present variations in S or P as B increases. This last suggestion was confirmed, for Zeeman contribution different from zero, by an explicit calculation of the GS vs B , which turns out to be always full spin and isospin polarized. Fig.4 displays the total energy of the GS as a function of the magnetic field. The arrows point to the places where the angular momentum jumps from one magic value to the next one, leaving the spin and parity unchanged. In the inset we show $E_{GS} - \beta N$ in order to compare with other publications which omit the N -dependent term. The nearly monotonous function of B is due to the fact that, in the absence of spin or isospin transitions, the interaction energy has a negligible influence in the plot and hence the evolution of the system is driven by the monotonous increasing term βN which is much more important than the decreasing term αM , which would produce kinks at the transition points, as it is shown in the inset ($E_{GS} - \beta N$ vs B). In brief, the full spin and isospin polarization appears to be a well defined attribute of these IS's that result from many-body effects.

As shown in Fig. 3A the interaction energy appears to be degenerated at the magic values with respect to the three possible spin polarizations. Since the curves that belong to $S = \frac{3}{2}$ and $\frac{1}{2}$ have lower energy at the end of the plateaux, the final balance of energy depends critically on the relation between this difference of interaction energy, the kinetic and the Zeeman terms. That is to say, an IS that is the GS for a given value of B and ω_0 will remain as GS as B increases or ω_0 decreases only if

$$E_{C+T}(M, \frac{3}{2}, 1) - E_{C+T}(M+1, \frac{3}{2}, 1) < \alpha + \Delta_Z \quad (12)$$

or otherwise the new GS will be a compressible not fully spin polarized state at $M + 1$. Hence, as B or ω_0 change, the GS can be driven into compressible zones in contrast with the results obtained for single QD's.

The single particle occupancies of the m -values for the first three IS's for $N = 5$, calculated at $B = 4, 7$ and 9 T respectively are shown in Fig.5. The first GS for $M = 10$ ($\nu = 1$) is the compact full polarized state which belongs to a one dimensional subspace and, as a consequence, no correlation is involved as one Slater determinant produces the

exact solution. Moreover, the density is a "dome" shape circular symmetric distribution without any structure. As B grows, the angular momentum changes from M to $M + N$, all the electrons jump together moving away from the origin and forming a ring. The dimension of the GS subspace increases and as the relative weights of the different Slater determinants within the expansion of the GS become significant for different elements of the bases, namely, the correlation becomes important.

C. DQD: Phase diagram Δ_t/d .

So far we have explored the situation $d \sim l_B$. In order to get the complete scenario of IS's in a DQD, we have also investigated in detail the remaining regions of the phase diagram (Δ_t / d)

In Figs.6-9 we follow, for $N = 3$, the variation of the interaction energy vs M ($E(C + T)/M$) as d and Δ_t change.

From A to B (Fig.6), as it was just discussed for the case $d \sim l_B$, the plateaux emerge as Δ_t grows from zero until they are well defined at $\Delta_t = 0.11 u$, that is to say, for values of M given by Eq.(11) the system evolves from compressible to incompressible states. For d close to zero (at the left of point B) and Δ_t large, all the electrons are in the symmetric state. As a consequence, the interaction energy of a single QD can be reproduced with high accuracy by the addition of the constant contribution $\Delta_t X/2$ to the energy of the DQD in this region.

From A to D (Fig.7), tunneling between the two dots is not allowed. Starting from a curve without plateaux for small distances ($d \sim 10\text{\AA}$), we move across the transition regime with a gradual formation of new plateaux at $M = M_R + M_L$ where M_R and M_L are the magic numbers of single QD's. We come close to the point D at $d = 1000\text{\AA}$ which show the features of two decoupled dots with $N = 1$ (with no contribution to the Coulomb term) and $N = 2$ (with magic numbers $M = 1, 3, 5, 7, \dots$) respectively. This regime in which tunneling is forbidden has been previously studied for double layers and special attention has been devoted to the $\nu = 1$ case^{7,8,12}. For a double layer the incompressible state $\nu = 1$ is observed for values of d about the magnetic length⁸. Furthermore, as d increases, the state exhibits a phase transition to a compressible one. The difference between the two cases comes from the fact that, as it was previously discussed, when $\Delta_t \sim 0$ M_R and M_L are well defined quantum numbers, the increase by one unit of M means the increase of M_R or M_L (but not both) changing, in a DQD, the relative position of the electrons in each dot and so decreasing the Coulomb interaction which prevents the formation of a plateau. However, this is not the case for a double layer in which the shift of charge due to the change of angular momentum does not change the relative inter-layer distribution of charge, allowing for the appearance of plateaus. For large d the two layers decouple and hence one would expect (for total $\nu = 1$) two $\nu = 1/2$ IS's. However, since fractional QHE of $\nu = 1/2$ is not observable for a single layer, these states were not identified in Ref.⁽⁸⁾.

From B to C (Fig.8), although the distance d grows, the sequence of magic numbers typical of a DQD does not disappear due to the relative large value of Δ_t ($\Delta_t = 0.229 u$).

An exceptional case appears for $d > 500\text{\AA}$ at $M = 1$ (for $N = 5$ the analog exceptional case appears at $M = 4$). It turns out to be the only IS within the LLL regime which does not fulfill the general rule of being full spin and isospin polarized. The subspace associated to the appropriate configuration, i.e., $(M = 1, S = 3/2, P = 1)$ is one dimensional and the only Slater determinant in the bases has one electron in the symmetric state and two electrons in the antisymmetric state (or $X = -1$). It is the only IS with no single dot analog. Our interpretation of the fact that the $M = 1$ magic value appears only for relative large distances is as follows: for large tunneling and small d , the system is closer to a single dot with $N = 3$ than to a DQD of the same number of electrons (as discussed before). Thus $M = 1$ can only appear when the Coulomb inter-dot interaction weakens related to Δ_t and DQD properties different from those of a single QD may arise. Notice however, that for the $M = 1$ state to be a GS such a low magnetic field (or large confining potential) is required that the LLL regime assumption would not apply anymore. Finally, even for values of d as large as 1500\AA (being $l_B = 65\text{\AA}$), we did not find the transition from DQD to two decoupled single dots.

From D to C (Fig.9), the tunneling increases and the system of two decoupled QD's with a period of two typical of the $N = 2$ single dot evolves into a DQD reproducing the period of three typical of a $N = 3$ DQD. During the transition, there is a narrow interval of values of Δ_t for which the $E(C + T)$ vs M curve has no plateaux (except for the $M = 1$ case). That is to say, an initially incompressible GS would evolve into a compressible state and again into a IS as Δ_t increases. This evolution takes place as the system changes from two decoupled single dots to a DQD.

Compressible regions have been obtained before by Rontani et al.¹⁶ for DQD with finite width. They consider the evolution of the GS of the system of $N = 6$ as d increases for Δ_t exponentially decreasing with d , which is equivalent to the evolution along a trajectory from B to D in our phase diagram. They obtain a small zone of compressibility in the middle, related to the transition from a regime where the system behaves as a unique coherent system to a regime of well separated QD's. We observe the same qualitative behavior along the B - C trajectory (which is different to theirs) although we do not obtain the same magic values.

The transition from DQD to two decoupled single QD's as Δ_t decreases has been observed before by Peeters et al¹⁰ by means of a current spin density functional calculation.

We have also studied the D to C evolution for $N=5$ which shows the same qualitative behavior. As D is approached the structure of plateaux can be understood in terms of the IS of two decoupled $N = 2$ and $N = 3$ single QD's, although the analysis is much more intricate than for the $N=3$ case.

IV. Discussion

We have investigated in detail the existence of IS's that result from the Coulomb many-body effects in a DQD for the entire phase diagram (Δ_t / d) .

An important point in our analysis is the criterium used to identify IS's. First we want to emphasize that in contrast to the case of a single layer for which the integer QHE is associated to gaps of single-particle origin and the fractional QHE is associated to gaps involving many-body effects, for double layered systems (and thus in accordance for DQD), single-particle as well as many-body regimes can be related to QHE at the same filling factor by the tuning of appropriate sample parameters⁸. The IS's we are interested in are those associated with e - e interaction (coupled with tunneling) and thus signaled somehow in the variation of the interaction energy with M . As discussed in the previous section, we define the interaction energy as the Coulomb plus tunneling contributions, and require IS's to preserve the interaction energy when the angular momentum M is increased by one unit. We want to stress that this is not equivalent to identifying magic M from the kinks of the lowest energies of each configuration as a function of M or from the kinks of the variation of the absolute GS energy as a function of B , as it has been used in the literature^{17,18} to identify correlated IS's.

Our criterium is equivalent to the one used by Laughlin in Ref.⁽¹⁾ for the single layer, as we discuss below. The Hamiltonian is separable into the CM and the relative coordinates and as a consequence, the total angular momentum can be analyzed as $M = M_{CM} + M_{rel}$ and the energy as $E_{tot} = E_{CM} + E_{internal}$. For three two dimensional electrons, Laughlin obtains that the *internal* energy as a function of the *relative* angular momentum has downward cusps at special (magic) values ($M = 3, 6, 9, 12, \dots$). These magic values appear to be related to incompressibility: the area of the system defined as the area of the triangle determined by the correlated positions of the electrons within these states changes discontinuously as pressure is applied. At the downward cusps, $E_{internal}(M_{rel}) < E_{internal}(M_{rel} + 1)$ for the lowest energy states of each configuration. They are the only states for which the increase of M_{rel} by one unit requires a positive amount of internal energy. In order to show that our criterium¹⁴ is equivalent to Laughlin's¹, notice first that $E_{C+T}(M)$ only depends on M_{rel} and $M_{rel} \leq M$. Since $E_{C+T}(M)$ is defined as the minimal energy among those of the states with total angular momentum M , it implies it is the minimal energy among all states with relative angular momentum $M_{rel} \leq M$. Hence, given M and $E_{C+T}(M - 1)$, $E_{C+T}(M - 1) \neq E_{C+T}(M)$ implies $E_{C+T}(M - 1) > E_{C+T}(M)$ and furthermore $M = M_{rel}$. Since E_{C+T} reduces to $E_{internal}$ for a single layer, if $E_{C+T}(M - 1) > E_{C+T}(M)$ then $E_{internal}(M_{rel} - 1) > E_{internal}(M_{rel})$ for $M_{rel} = M$. Namely, negative slopes in our plots imply negative slopes in Laughlin's. If, on the contrary, $E_{C+T}(M - 1) = E_{C+T}(M)$, then M_{rel} has not changed and M contains at least one unit of CM angular momentum. Being $E_{C+T}(M)$ the minimal energy with total angular momentum M , it implies that any state with $M_{rel} = M$ has larger energy than $E_{C+T}(M - 1)$. If, in addition, $E_{C+T}(M - 2) > E_{C+T}(M - 1)$, as it is always the case in our plots, then $M - 1$ contains only relative angular momentum, as shown above. Then, for the single layer, $E_{C+T}(M - 1) = E_{C+T}(M)$ implies $E_{internal}(M_{rel} - 1) < E_{internal}(M_{rel})$ for $M_{rel} = M$. Namely plateaux in our plots imply positive slopes in Laughlin's, which concludes our proof. On the other hand, there is no guarantee that downward cusps in the curve E_{tot} vs M are related to plateaux in the curve E_{C+T} vs M . It is enough to have $\alpha > E_{C+T}(M) - E_{C+T}(M + 1)$ at a non-magic M to obtain there a downward peak in E_{tot} vs M which is not related to a plateau in E_{C+T} vs M .

For $d \sim l_B$ the fact that the magic M follow Eq.(11) is in conflict with the claim made in Ref.^{17,18} that extra magic values for M (depending on the value on the tunneling strength) exist in this regime. The authors of Ref.^(17,18) identified IS's with downward cusps of the total energy as a function of M , i.e. the interaction energy (our curve) with the addition of the single-particle contribution. In Fig.10 we show the two possibilities for $N = 5$. It is clear from the upper curve that some downward cusps, which would be identified as IS's by the criterium of Ref.^{17,18}, do not actually correspond to IS's in our criterium.

In order to make sure that the discrepancies with Ref.^{17,18} are only due to the different criteria to identify IS's, we have reproduced their results (see Fig.11). To be more precise, we performed the calculation for $N = 3$ and the same input parameters as those used in Ref.¹⁸ ($N = 3$, $B = 15 T$, $d = 200 \text{ \AA}$, $\hbar\omega_0 = 3 \text{ meV}$ and $\Delta_t = 0.2 \text{ meV}$). In Fig.11A the interaction energy contribution versus M is shown. Due to the low value of the tunneling contribution, the plateaux that will appear, for larger values of Δ_t , at $M = 3$, 6 and 9 are still not visible and the only ones that already appear are $M = 12$ and 15. If the kinetic contribution is added, it comes out that the GS is at $M = 5$ (see Fig.11B) at the lowest downward cusp of the total energy in accordance with Ref.¹⁸.

A word of caution should be given here as a number of relevant papers exists in the literature^{18–21} which use the term *magic angular momentum* to denote the angular momentum M which displays downward cusps in the curve $E_{tot}(M)$. The corresponding states enjoy enhance stability and have been the subject of intensive studies. In particular Refs. ^(19,20) provide characterizations of these states ranging from small values of M , where the fractional quantum Hall regime is sometimes identified, to large values of M , where strip-like structures and Wigner molecules seem to appear (see Ref. ⁽²¹⁾ for a review and Ref. ⁽²²⁾ for related work on layers). However, only a subset of these states fulfills our criterium of incompressibility and the magic M displayed in (11) correspond to this subset only.

Our conclusions can be summarized as follows:

(i) The downward cusps obtained by Laughlin in Ref.⁽¹⁾ turn out to be equivalent to the plateaux of the curve E_{C+T} vs M .

(ii) All the incompressible states are full spin and isospin polarized (except the $M = 1$ case for $N = 3$). Since a single QD full spin polarized and a DQD full spin and isospin polarized are systems with no extra degrees of freedom aside from angular momentum, we expect a similar behavior for the electronic distance quantization as that obtained in Ref⁽¹⁾.

(iii) An exceptional incompressible state was found for $d/l_B \geq 8$ at $M = 1$ and $X = -1$ for $N = 3$ and $M = 4$ and $X = 1$ for $N = 5$. This is the only one that is not full isospin polarized and does not have a single QD analog. However, for it to be a GS values of the input parameters that do not fulfill the assumption of the LLL regime are required.

(iv) For $d \sim l_B$, it is not possible to obtain IS's if the tunneling is small.

(v) For small Δ_t , as d grows, the DQD evolves into two decoupled single QD's. New magic values appear which correspond to the addition of magic numbers of two decoupled QD's.

(vi) An IS at M will remain as the GS under changes of B or/and ω_0 if the condition

$$E_{C+T}(M) - E_{C+T}(M+1) < \alpha + \Delta_Z \quad (13)$$

is fulfilled. In general, however, the variation of B , ω_0 or d can drive the GS from incompressible into compressible zones of the phase diagram. This behavior differs from the well known properties of a single QD for which the variation of the GS as B or ω_0 change is driven through incompressible states only, skipping all non-magic values of M .

(vii) Whenever the GS is full spin and isospin polarized, it is an IS. In other words, the IS's are the only possible GS for ν lower than one. However for $\nu > 1$ other GS are possible.

Before closing, let us briefly elaborate on the last point. Notice that the following situations are also possible: (a) GS's with M that fulfills Eq. (11) and are not IS's due to the fact that from M to $M+1$ there is not a plateau. This condition can be obtained for very low values of Δ_t , for example, for $N = 5$, $M = 10$, $B = 5T$, $\hbar\omega_0 = 2.6 \text{ meV}$, $\Delta_t = 0.2 \text{ meV}$ and $d = 20 \text{ \AA}$. (b) Ground

states with M not given by Eq. (11) for which the system is not full spin or isospin polarized, (or $\nu > 1$) and $E(C+T)$ has not a constant evolution from M to $M+1$. This is the case for example for $N = 5$, $M = 13$ ($B = 6T$, $\hbar\omega_0 = 2.6 \text{ meV}$, $d = 20 \text{ \AA}$ and $\Delta_t = 5.86 \cdot 10^{-3} \text{ meV}$). In the last case, $S = N/2$ and $P = -1$, however $X \neq N$.

Notice also that the previously discussed states are not the only possible GS's within the LLL regime. For instance, if the confining potential is strong enough, other types of GS's are possible like the ferromagnetic, canted and symmetric states (all of them with $\nu = 2$) first studied in double layers²³ and latter recognized in DQD²⁴.

Finally, let us note that the correlation plays an increasingly important role as the magnetic field grows up. The interaction energy and correlation effects can be experimentally tested by uniform electric fields with non-vanishing component along the z-direction due to the fact that under this condition, the Kohn theorem does not apply¹³ and the FIR spectroscopy becomes sensitive to the internal structure.

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FIGURES

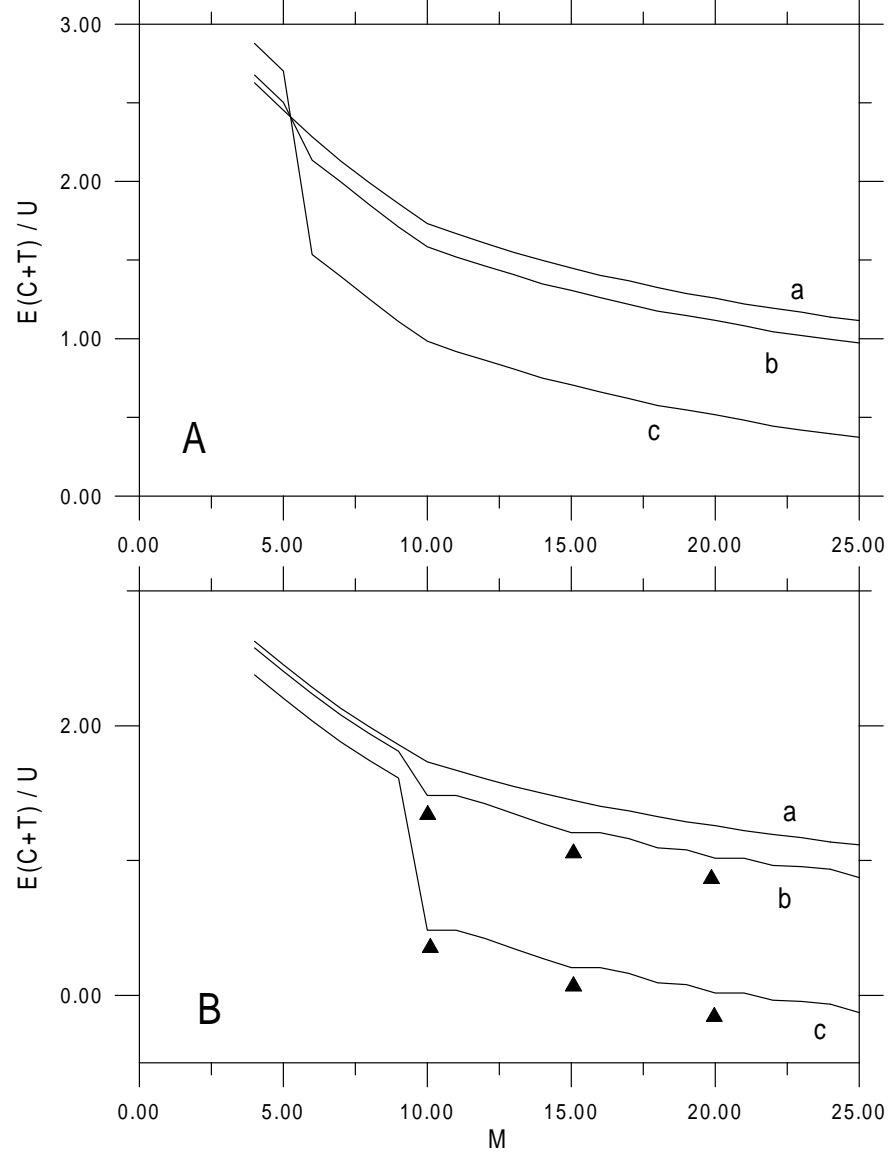


FIG. 1. A: Coulomb plus tunneling contribution to the total energy as a function of M for $N = 5$, $S = N/2$ and parity $P = 1$, for several values of the tunneling gap: (a) $\Delta_t = 0$, (b) $\Delta_t = 2.2 \text{ meV}$ and (c) $\Delta_t = 11 \text{ meV}$. B: The same as A for $P = -1$. The triangles point to the beginning of the plateaux. We have taken $B = 5T$, $d = 20 \text{ \AA}$, and $\hbar\omega_0 = 2.6 \text{ meV}$.

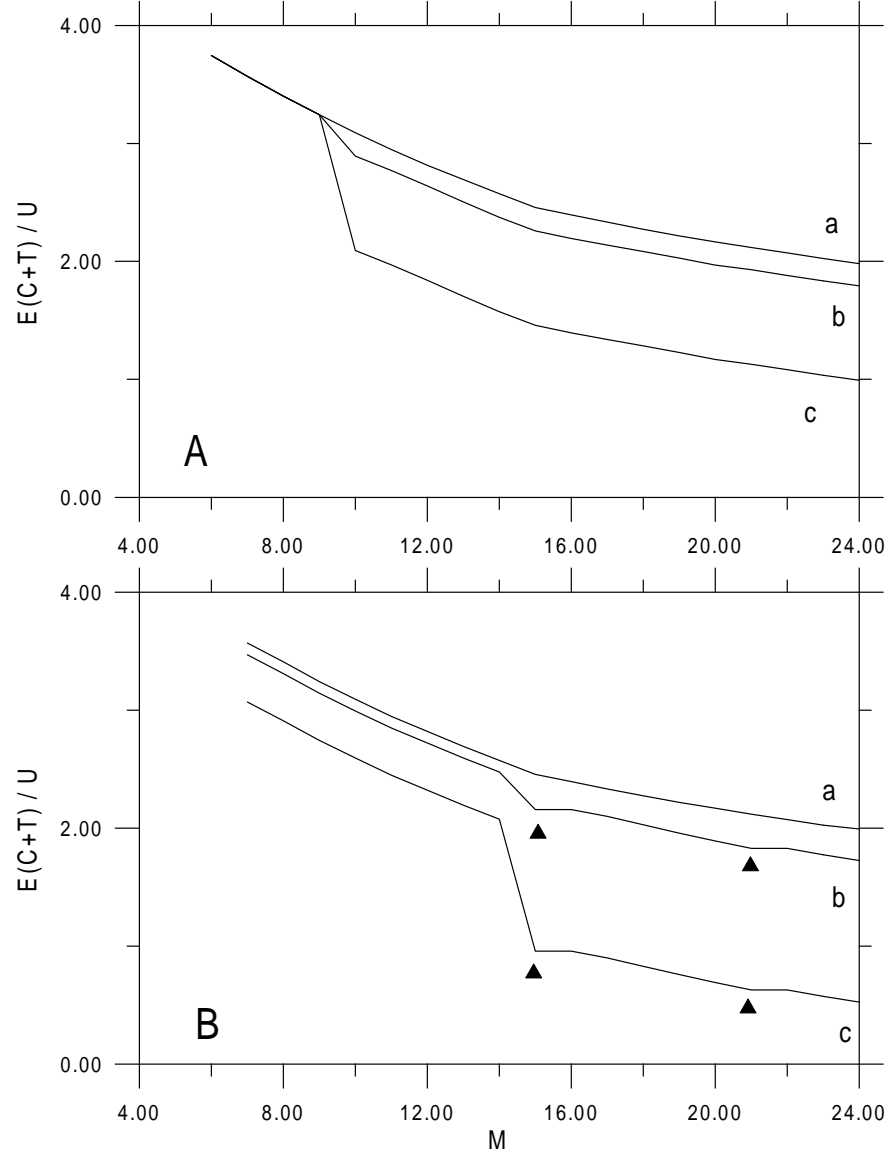


FIG. 2. The same as Fig.1 for $N = 6$

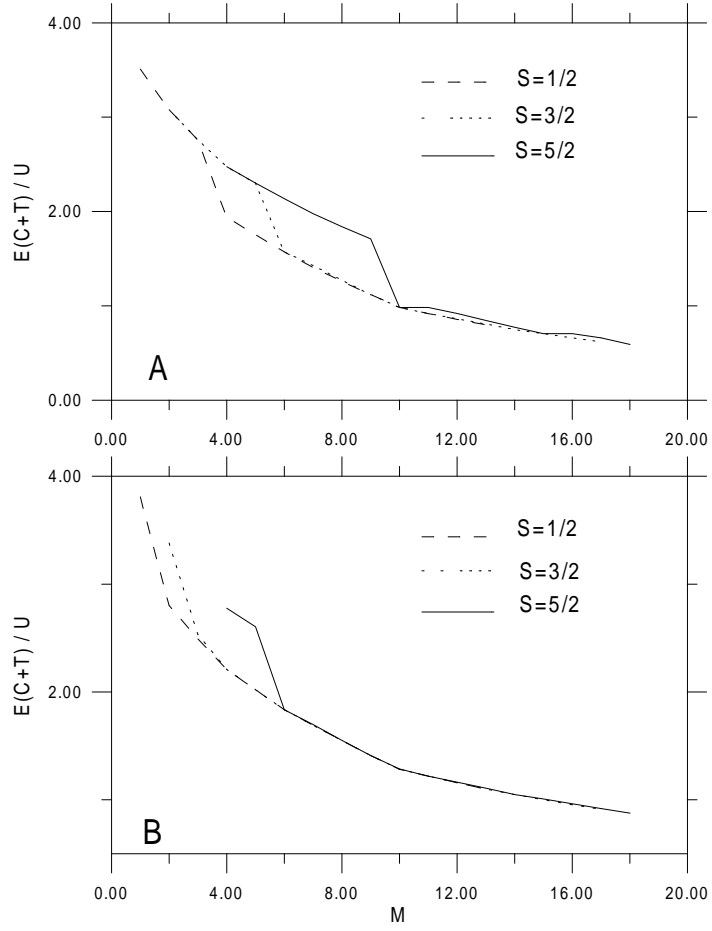


FIG. 3. A: Coulomb plus tunneling contribution to the total energy as a function of M for $N = 5$ and parity $P = -1$ for all the possible values of the spin S . B: The same as A for $P = 1$. We have taken $B = 5T$, $d = 20\text{\AA}$, $\hbar\omega_0 = 2.6\text{ meV}$ and $\Delta_t = 2\text{ meV}$.

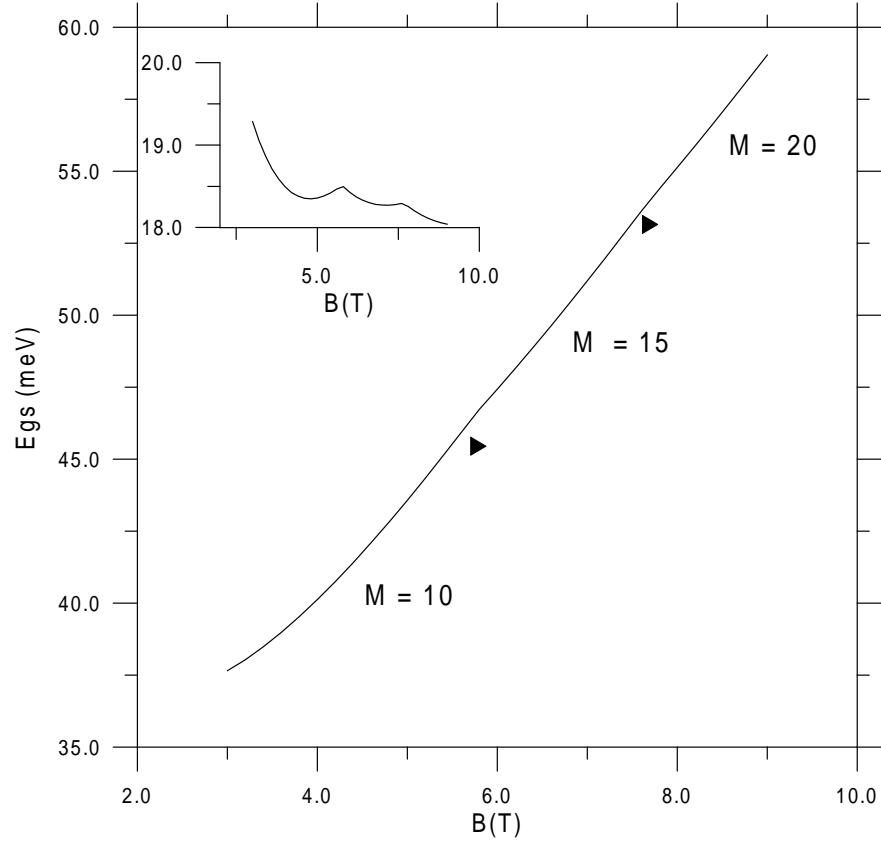


FIG. 4. A: Ground state energy versus B for $N = 5$. The triangles point to the transitions between magic angular momenta. We use the same values for d , ω_0 and Δ_t as in Fig.3. Inset: $E_{GS} - \beta N$ as a function of B .

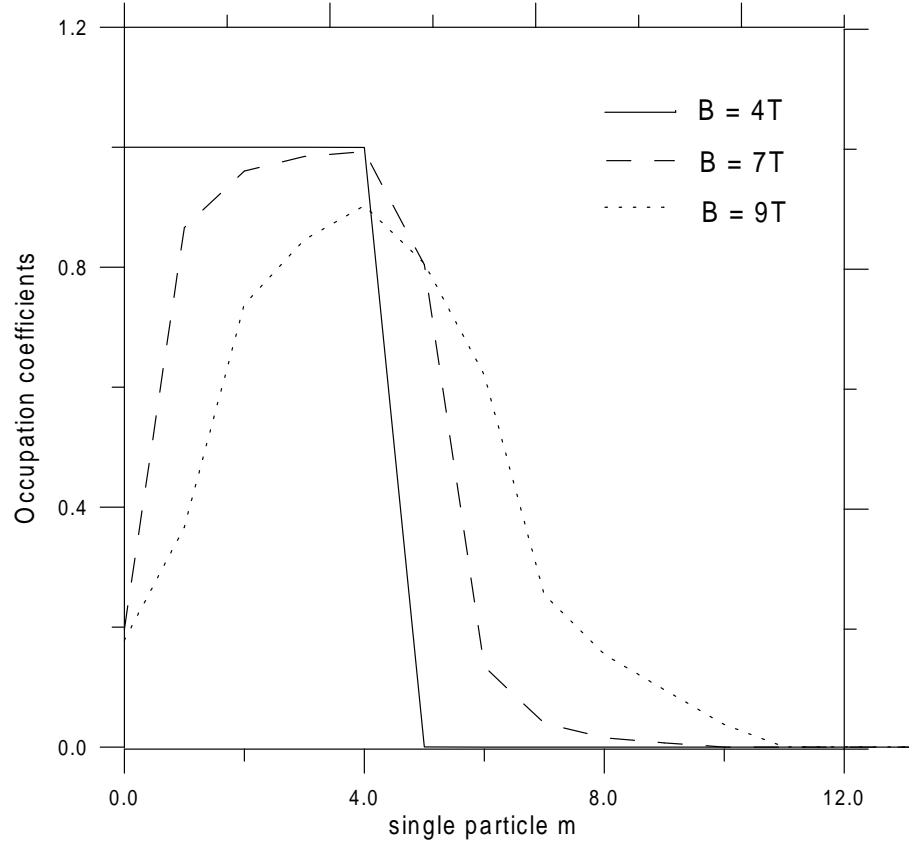


FIG. 5. Total occupancy of the single-particle angular momentum states m for $N = 5$. The values of B considered correspond to the magic values $M = 10, 15$ and 20 respectively. We have taken $d = 20\text{\AA}$, $\hbar\omega_0 = 2.6\text{ meV}$ and $\Delta_t = 6\text{ meV}$.

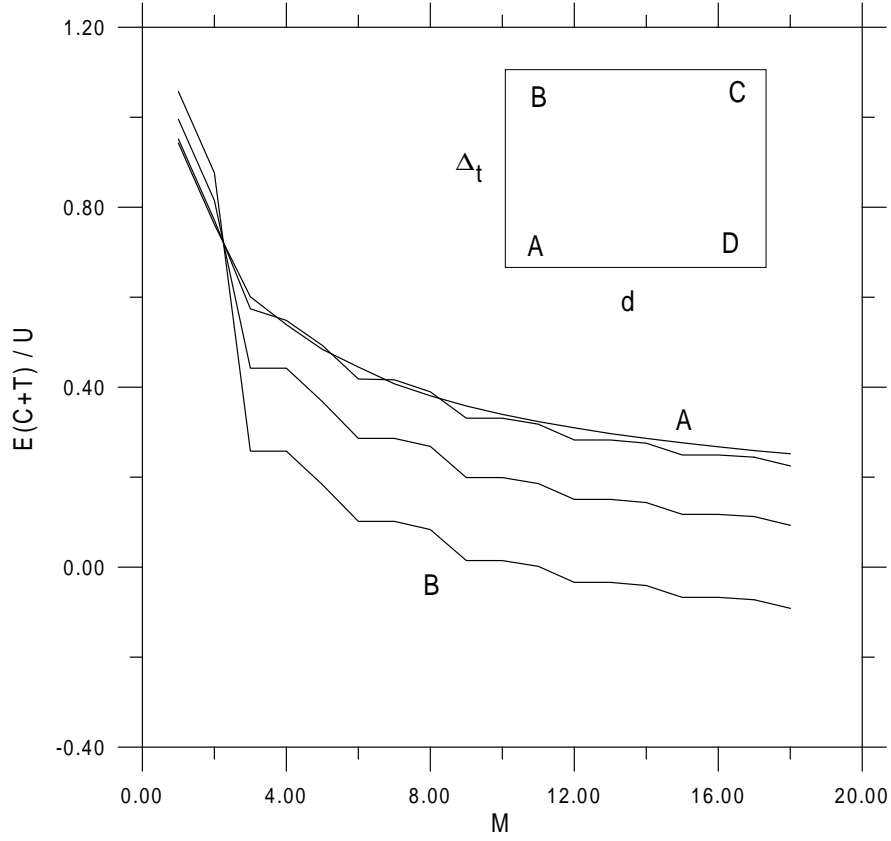


FIG. 6. Coulomb plus tunneling contribution versus M for $N = 3$. From A to B the tunneling gap is: 0.0, 0.018, 0.106 and $0.229u$ respectively. Inset: phase diagram used. We have taken $\hbar\omega_0 = 2.6\text{ meV}$, $B = 15\text{ T}$, $d = 10\text{ \AA}$, $S = 3/2$ and $P = 1$ ($l_B = 65\text{ \AA}$).

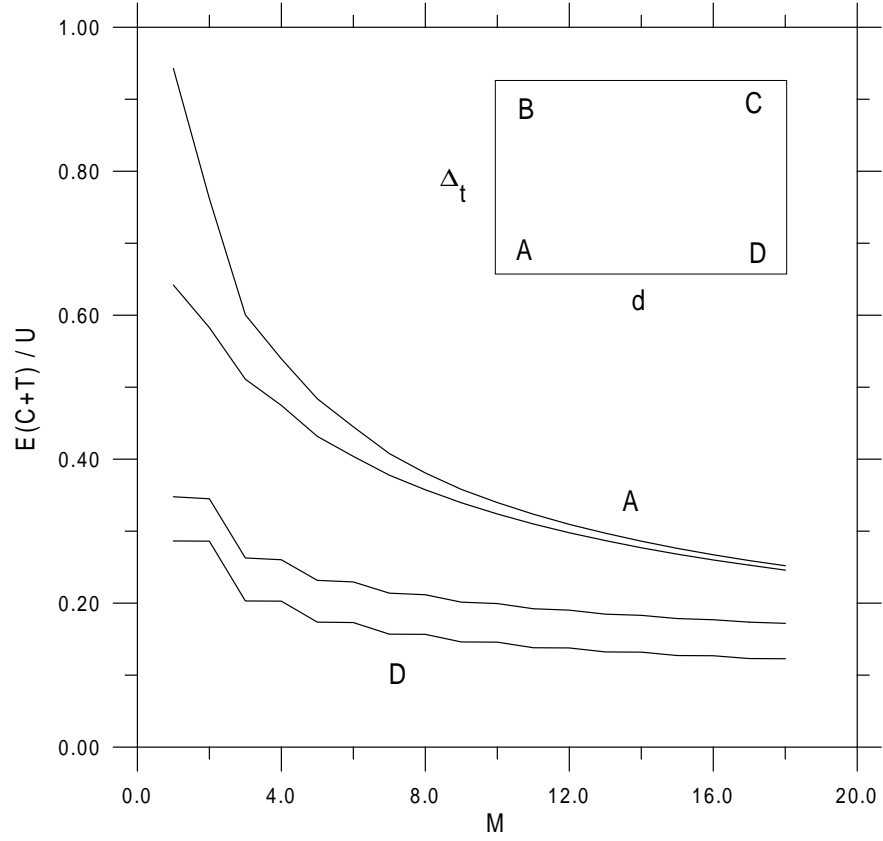


FIG. 7. The same as Fig.6 for $\Delta_t = 0$. From A to D the inter-dot distance is: 10, 50, 500 and 1000 Å respectively. Inset: phase diagram used.

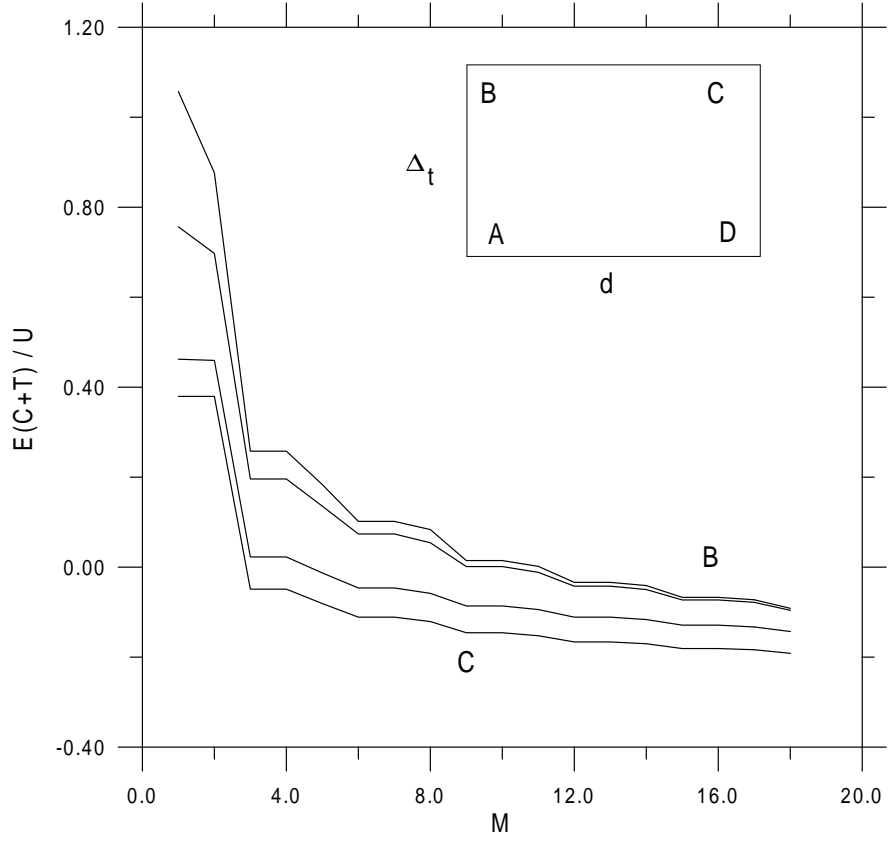


FIG. 8. The same as Fig.6 for $\Delta_t = 0.229 u$. From B to C the inter-dot distance is: 50, 100, 500 and 1000 Å respectively. Inset: phase diagram used.

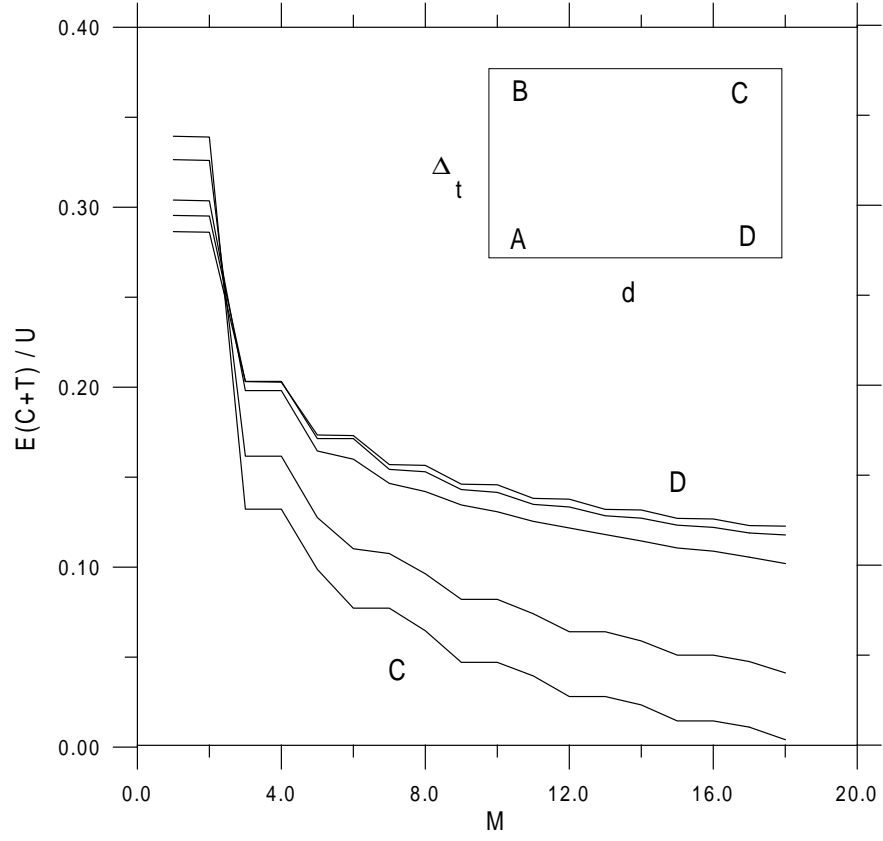


FIG. 9. The same as Fig.6 for $d = 1000 \text{ \AA}$. From D to C the tunneling gap is: 0.0, 0.018, 0.035, 0.080 and 0.106 u respectively. Inset: phase diagram used.

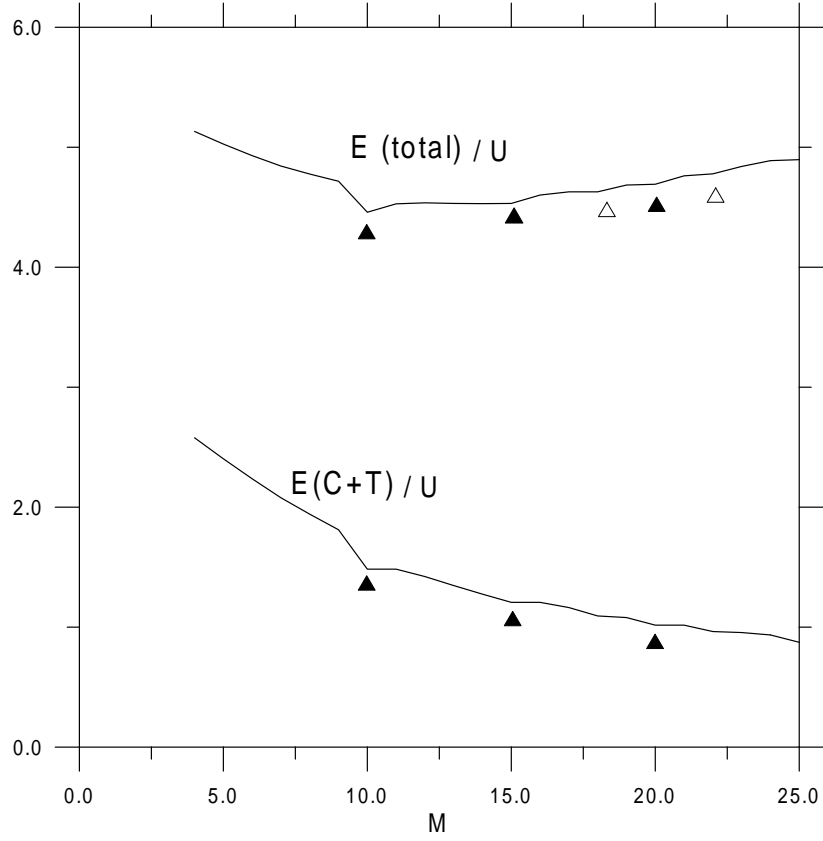


FIG. 10. Coulomb plus tunneling contribution (lower curve) and total energy (upper curve) as a function of M for $N = 5$. The black triangles point to the actual magic values M whereas the white triangles point to cusps which could be mistaken by them. The values of B , ω_0 , d and Δ_t are the same as in Fig.3

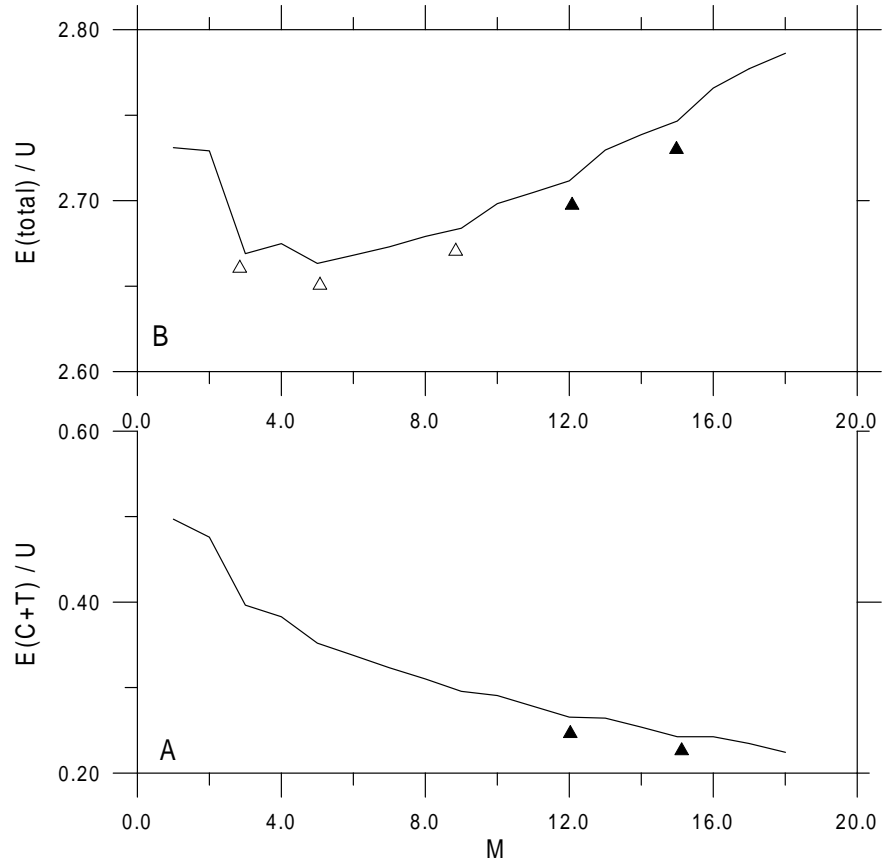


FIG. 11. A: Coulomb plus tunneling contribution versus M for $N = 3$. B: The same as in A for the total energy. The input parameters are given in the text. The black triangles point to the actual magic values M whereas the white triangles point to cusps which could be mistaken by them.